Algebra I

M. Math.

8 September 2015

All rings are assumed to be commutative with identity. The rings of integers, rational numbers, real numbers and complex numbers are denoted by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ respectively.

All questions carry equal (non-zero) marks.

1. Let $\phi : R \to S$ be a surjective ring homomorphism with kernel K. Prove that there is a bijective correspondence between the set of all ideals in S and the set of ideals in R that contain K.

2. Let a be an element in a ring R. Let S be a ring obtained from R be adjoining an inverse of a. Prove that S is a zero ring if and only if a is *nilpotent* in R.

3. Suppose S is a ring obtained from the ring of real numbers $\mathbb{R}$ by adjoining and element α satisfying the relation $\alpha^2 = 1$. Prove that S is isomorphic to the product ring $\mathbb{R} \times \mathbb{R}$.

4. Prove that two integer polynomials are relatively prime in the ring $\mathbb{Q}[X]$ if and only if the ideal generated by them in $\mathbb{Z}[X]$ contains a non-zero integer.

5. Let S be a subring of the polynomial ring $\mathbb{C}[t]$ which strictly contains $\mathbb{C}$. Prove that $\mathbb{C}[t]$ is a finitely generated S module.